JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 2, May 2016
http://dx.doi.org/10.14403/jcms.2016.29.2.221

HALF b - CONNECTEDNESS IN TOPOLOGICAL SPACES

Takashi Noiri* and Shyamapada Modak**

Abstract. This paper will discuss the huge change of b - separated, b - connectedness, b - disconnectedness and their applications when the definition of b - separated and b - connectedness are small changed.

1. Introduction and preliminaries

The idea of b - open sets has been introduced by D. Andrijević [1] in 1996, but the same is defined by El-Atik [6] under the term of γ - open sets. Formally a set A in a topological space X is said to be b - open if $A \subset Cl(Int(A)) \cup Int(Cl(A))$, where '$Cl$' and '$Int$' denote the closure and interior operator respectively in the space X. The collection of all b - open sets in a space X is denoted as $BO(X)$. The complement of a b - open set is called a b - closed set. The b - closure of a set A, denoted by $bCl(A)$, is the intersection of all b - closed sets containing A. $bCl(A)$ is the smallest b - closed set containing A. The b - interior of a set A, denoted by $bInt(A)$, is the union of all b - open sets contained in A. $bInt(A)$ is the largest b - open set contained in A.

In the theory of b - open sets, connectedness and disconnectedness [7] have already been defined. In this paper, we define and investigate the notions of half b - separated sets and half b - connected sets with the help of b - open sets in a topological space. We shall draw the interrelation between half b - connectedness (resp. half b - separated sets) and b - con nectedness (resp. b - separated sets). We shall also discuss the situation

Received February 06, 2015; Accepted April 26, 2016.
2010 Mathematics Subject Classification: Primary 54A05, 54D05.
Key words and phrases: Half b - separated, Half b - connected, b - irresolute, γT_0 - space.
Correspondence should be addressed to Shyamapada Modak, spmodak2000@yahoo.co.in.

of the b - irresolute, b - continuous and b - closed image of half b - connected spaces.

2. Half b - separated sets

DEFINITION 2.1. Two subsets A and B in a space X are said to be half b - separated (resp. b - separated [4], $Cl-Cl$ - separated [8]) if and only if $A \cap bCl(B) = \emptyset$ or $bCl(A) \cap B = \emptyset$ (resp. $A \cap bCl(B) = \emptyset = bCl(A) \cap B$, $Cl(A) \cap Cl(B) = \emptyset$).

DEFINITION 2.2. (i) [7] A subset S of a space X is said to be b - connected relative to X if there are no two b - separated subsets A and B relative to X with $S = A \cup B$.

(ii) A subset A of a space X is said to be half b - connected (resp. $Cl-Cl$ - connected [8]) if A is not the union of two nonempty half b - separated (resp. $Cl-Cl$ - separated) sets in X.

REMARK 2.3. From the above definitions, we have the following implications. However, converses are not always true as shown in the following examples.

$$Cl - Cl - separated \Longrightarrow separated \Longrightarrow b - separated \Longrightarrow$$
$$half\ b - separated$$

EXAMPLE 2.4. *Let* $X = \{a, b, c, d\}$ *with a topology* $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. $BO(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. *$b$ - closed sets are:* $X, \emptyset, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{c, d\}, \{b, d\}, \{a, d\}, \{d\}, \{c\}, \{b\}$ *and* $\{a\}$. *Here* $\{a, b\}$ *and* $\{c, d\}$ *are half b - separated as* $(\{a, b\}) \cap bCl(\{c, d\}) = \emptyset$. *Since* $bCl(\{a, b\}) \cap (\{c, d\}) \neq \emptyset$, *so they are not b - separated.*

From the fact that $bCl(A) \subset Cl(A)$ for every subset A of X, every $Cl-Cl$ - separated set is *half b - separated.* But the converse may not be true as shown in the following example.

EXAMPLE 2.5. *Consider Example 2.4, the sets* $\{a, b\}$ *and* $\{c, d\}$ *are half - b - separated. Now* $Cl(\{a, b\}) \cap Cl(\{c, d\}) \neq \emptyset$. *So they are not* $Cl - Cl$ *- separated.*

EXAMPLE 2.6. [7] *Let* $X = \{a, b, c, d\}$ *with a topology* $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. *The subsets* $\{a\}$, $\{c, d\}$ *are b - separated but not separated.*

EXAMPLE 2.7. [8] *In* $\mathbb{R}$ *with the usual topology on* $\mathbb{R}$ *the sets* $A = (0, 1)$ *and* $B = (1, 2)$ *are separated sets but not* $Cl-Cl$ *- separated sets.*

Theorem 2.8. *Let A and B be nonempty sets in a space X. The following statements hold:*

(i) *If A and B are half b - separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are so.*
(ii) *If $A \cap B = \emptyset$ and one of A and B is b - closed or b - open, then A and B are half b - separated.*
(iii) *If one of A and B is b - closed or b - open and if $H = A \cap (X - B)$ and $G = B \cap (X - A)$, then H and G are half b - separated.*

Proof. (i) This is obvious.

(ii) In case A is b - open by $A \cap B = \emptyset$, $A \cap bCl(B) = \emptyset$. In case A is b - closed $bCl(A) \cap B = A \cap B = \emptyset$. Therefore, A and B are half b - separated. In case B is b - open or b - closed it follows similarly that A and B are half b - separated.

(iii) (1) Let A be b - closed (B be b - closed), then we have

$bCl(H) \cap G \subset bCl(A) \cap (X - A) = A \cap (X - A) = \emptyset$ ($H \cap bCl(G) \subset (X - B) \cap bCl(B) = \emptyset$).

(2) Let A be b - open (B be b - open). Then we have

$H \cap bCl(G) \subset A \cap bCl(X - A) = A \cap (X - A) = \emptyset$ ($bCl(H) \cap G \subset bCl(X - B) \cap B = \emptyset$). Therefore, H and G are half b - separated. □

Theorem 2.9. *The subsets A and B of a space X are half b - separated if and only if there exists U in $BO(X)$ such that $A \subset U$ and $B \cap U = \emptyset$ or there exists V in $BO(X)$ such that $B \subset V$ and $A \cap V = \emptyset$.*

Proof. Let A and B be half b - separated sets, then $A \cap bCl(B) = \emptyset$ or $bCl(A) \cap B = \emptyset$. Suppose $A \cap bCl(B) = \emptyset$. Set $U = X - bCl(B)$, then we have $U \in BO(X)$, $A \subset U$ and $B \cap U = \emptyset$. Suppose $bCl(A) \cap B = \emptyset$. Set $V = X - bCl(A)$. Then we have $V \in BO(X)$, $B \subset V$ and $A \cap V = \emptyset$.

Conversely, suppose that there exists $U \in BO(X)$ such that $A \subset U$ and $B \cap U = \emptyset$. Then $bCl(B) \cap U = \emptyset$ and hence $A \cap bCl(B) \subset U \cap bCl(B) = \emptyset$. Thus A and B are half b - separated. In case the another condition holds, the proof is similar. □

3. Half b - connected sets

Remark 3.1. The following implications follow from Remark 2.3.

$$half\ b - connected \Longrightarrow b - connected \Longrightarrow connected \Longrightarrow Cl - Cl - connected$$

THEOREM 3.2. *A space X is half b - connected if and only if it cannot be expressed as the disjoint union of a nonempty b - open set and a nonempty b - closed set.*

Proof. Let X be a half b - connected space. If possible suppose that $X = U \cup F$, where $U \cap F = \emptyset$, $U(\neq \emptyset)$ is a b - open set and $F(\neq \emptyset)$ is a b - closed set in X. Since F is a b - closed set in X, then $U \cap bCl(F) = \emptyset$ and hence U and F are half b - separated. Therefore X is not a half b - connected space. This is a contradiction.
Conversely, suppose that X is not a half b - connected space, then there exist nonempty half b - separated sets A and B such that $X = A \cup B$. Let $A \cap bCl(B) = \emptyset$. Set $U = X - bCl(B)$ and $F = X - U$. Then $U \cup F = X$ and $U \cap F = \emptyset$. And also U is a nonempty b - open set and F is a nonempty b - closed set.
In case $bCl(A) \cap B = \emptyset$ we have the similar argument. □

Let recall the definitions of a γT_0 space and a γT_2 space .

DEFINITION 3.3. [5] A space X is said to be
(i) γT_0 if for each pair of distinct points in X, there exists a b - open set containing one of them but not the other,
(ii) γT_2 if for each pair of distinct points $x,\ y \in X$, there exist b - open sets U and V such that $x \in U,\ y \in V$ and $U \cap V = \emptyset$.

THEOREM 3.4. *Let X be a γT_0 space, where $/X/ \geq 2$, then it is not half b - connected.*

Proof. Let $x,\ y$ be distinct points of X. Then there exists a b - open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. In case $x \in U$ and $y \notin U$ it follows that $y \in X - U,\ X - U$ is b - closed. In case $x \notin U$ and $y \in U$ it follows that $x \in X - U$ and $X - U$ is b - closed. Furthermore, $X = U \cup (X - U)$, of course U and $X - U$ are disjoint. Therefore, by Theorem 3.2 X is not half b - connected. □

COROLLARY 3.5. *Let X be a γT_2 space, then it is not half b - connected.*

THEOREM 3.6. *Let X be a space. If A is a half b - connected subset of X and $H,\ G$ are half b - separated subsets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof. Let A be a half b - connected set. Let $A \subset H \cup G$. Since H and G are half b - separated, $G \cap bCl(H) = \emptyset$ or $bCl(G) \cap H = \emptyset$. Let $G \cap bCl(H) = \emptyset$. Since $A = (A \cap H) \cup (A \cap G)$, then $(A \cap G) \cap bCl(A \cap H) \subset$

$G \cap bCl(H) = \emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then A is not half b - connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. In case $bCl(G) \cap H = \emptyset$ we have the same argument. □

THEOREM 3.7. *If A and B are half b - connected sets of a space X and A and B are not half b - separated, then $A \cup B$ is half b - connected.*

Proof. Let A and B be half b - connected sets in X. Suppose $A \cup B$ is not half b - connected. Then, there exist two nonempty half b - separated sets G and H such that $A \cup B = G \cup H$. Since G and H are half b - separated, $G \cap bCl(H) = \emptyset$ or $bCl(G) \cap H = \emptyset$. Suppose that $G \cap bCl(H) = \emptyset$. Since A and B are half b - connected, by Theorem 3.6, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$.
Case (i). If $A \subset G$ and $B \subset H$, then $A \cap bCl(B) \subset G \cap bCl(H) = \emptyset$.

Thus, A and B are half b - separated, which is a contradiction. Hence, $A \cup B$ is half b - connected.
Case (ii). If $B \subset G$ and $A \subset H$, then $bCl(A) \cap B \subset bCl(H) \cap G = \emptyset$.

Thus, A and B are half b - separated, which is a contradiction. Hence, $A \cup B$ is half b - connected.

In case $bCl(G) \cap H = \emptyset$ we have the similar argument. □

THEOREM 3.8. *If $\{M_i : i \in I\}$ is a nonempty family of half b - connected sets of a space X and $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is half b - connected.*

Proof. Suppose $\cup_{i \in I} M_i$ is not half b - connected. Then we have $\cup_{i \in I} M_i = H \cup G$, where H and G are nonempty half b - separated sets in X. Since $\cap_{i \in I} M_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 3.6, $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\cup_{i \in I} M_i \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that H is empty. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is half b - connected. □

THEOREM 3.9. *Let X be a space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of half b - connected sets and A be a half b - connected set. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \triangle$, then $A \cup (\cup_{\alpha \in \triangle} A_\alpha)$ is half b - connected.*

Proof. Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 3.8, $A \cup A_\alpha$ is half b - connected for each $\alpha \in \triangle$. Moreover, $A \cup (\cup A_\alpha) = \cup(A \cup A_\alpha)$ and $\cap(A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 3.8, $A \cup (\cup A_\alpha)$ is half b - connected. □

Definition 3.10. (i) [4, 6] A function $f : X \to Y$ is said to be

(1) b - continuous if the inverse image of each open set in Y is b - open in X.

(2) b - closed if the image of each closed set in X is b - closed in Y.

(ii) [2] A function $f : X \to Y$ is said to be b - irresolute if for each point $x \in X$ and each b - open set V of Y containing $f(x)$, there exists a b - open set U of X containing x such that $f(U) \subset V$.

Theorem 3.11. *The b - irresolute image of a half b - connected space is half b - connected.*

Proof. Let $f : X \to Y$ be a b - irresolute function and X be a half b - connected space. If possible suppose that $f(X)$ is not a half b - connected subset of Y. Then there exist nonempty half b - separated sets P and Q in Y such that $f(X) = P \cup Q$. Since P and Q are half b - separated, $bCl(P) \cap Q = \emptyset$ or $P \cap bCl(Q) = \emptyset$. Since f is b - irresolute, we have $bCl(f^{-1}(P)) \cap f^{-1}(Q) \subset f^{-1}(bCl(P)) \cap f^{-1}(Q) = f^{-1}(bCl(P) \cap Q)) = \emptyset$ or $f^{-1}(P) \cap bCl(f^{-1}(Q)) \subset f^{-1}(P) \cap f^{-1}(bCl(Q)) = f^{-1}(P \cap bCl(Q)) = \emptyset$. Since $P \neq \emptyset$, there exists a point $p \in X$ such that $f(p) \in P$ and hence $f^{-1}(P) \neq \emptyset$. Similarly, we have $f^{-1}(Q) \neq \emptyset$. Therefore, $f^{-1}(P)$ and $f^{-1}(Q)$ are nonempty half b - separated sets such that $X = f^{-1}(P) \cup f^{-1}(Q)$.

Therefore X is not a half b - connected space. This is a contradiction. Hence $f(X)$ is a half b - connected space. □

Lemma 3.12. [6] *Let $f : X \to Y$ be a b - continuous function. Then $bCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

Theorem 3.13. *If $f : X \to Y$ is a b - continuous function and K is half b - connected in X, then $f(K)$ is $Cl - Cl$ - connected in Y.*

Proof. Suppose $f(K)$ is not $Cl{-}Cl$ - connected in Y. There exist two nonempty $Cl{-}Cl$ - separated sets P and Q of Y such that $f(K) = P \cup Q$. Set $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Since $f(K) \cap P \neq \emptyset$, then $K \cap f^{-1}(P) \neq \emptyset$ and so $A \neq \emptyset$. Similarly we have $B \neq \emptyset$. Moreover, we have $A \cup B = (K \cap f^{-1}(P)) \cup (K \cap f^{-1}(Q)) = K \cap (f^{-1}(P) \cup f^{-1}(Q)) = K \cap f^{-1}(P \cup Q) = K \cap f^{-1}(f(K)) = K$.

Case(i). Suppose $P \cap Cl(Q) = \emptyset$. Since f is b - continuous, then by Lemma 3.12, $A \cap bCl(B) \subset f^{-1}(P) \cap bCl(f^{-1}(Q)) \subset f^{-1}(Cl(P)) \cap f^{-1}(Cl(Q)) = f^{-1}(Cl(P) \cap Cl(Q)) = \emptyset$.

Case(ii). Suppose $Cl(P) \cap Q = \emptyset$. Now by Lemma 3.12, $bCl(A) \cap B \subset bCl(f^{-1}(P)) \cap f^{-1}(Q) \subset f^{-1}(Cl(P)) \cap f^{-1}(Cl(Q)) = f^{-1}(Cl(P) \cap Cl(Q)) = \emptyset$. This is contrary to that K is half b - connected. □

Corollary 3.14. *If $f : X \to Y$ is a bijective b - closed function and K is half b - connected in Y, then $f^{-1}(K)$ is $Cl - Cl$ - connected in X.*

Proof. Let $f : X \to Y$ be a b - closed bijection. Then $f^{-1} : Y \to X$ is a b - continuous bijection. Since K is half b - connected in Y, by Theorem 3.13, $f^{-1}(K)$ is $Cl - Cl$ - connected in X. □

References

[1] D. Andrijević, *On b - open sets,* Mat. Vesnik **48** (1996), 59-64.

[2] M. Caldas, S. Jafari, and R. M. Latif, *Applications of b - open sets and (b, s) - continuous functions,* King Fahd University of Petroleum and Minerals, Dept. of Math. Sci. Technical Report Series TR 400, 2008.

[3] E. Ekici, *On γ - normal spaces,* Bull. Math. Soc. Sci. Math. Roumanie **50** (**98**) (2007), no. 3, 259-272.

[4] E. Ekici, *On separated sets and connected spaces,* Demonstratio Math. **40** (2007), no. 1, 209-217.

[5] E. Ekici, On γ - *US* spaces, Ind. J. Math. **47** (2005), 131-138.

[6] A. A. El-Atik, *A study of some study of some types of mappings on topological spaces,* M. Sc. Thesis, Tanta University, 1997.

[7] A. A. Atik, H. M. Abu Donia, and A. S. Salama, *On b - connectedness and b - disconnectedness and their applications,* J. Egypt. Math. Soc. **21** (2013), 63-67.

[8] S. Modak and T. Noiri, *A weker form of connectedness*, Commun. Fac. Sci. Univ. Ank. Sr. A1 Math. Stat. **65** (2016), no. 1, 49-52.

*

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi
Kumomoto-ken 869-5142, Japan
E-mail: t.noiri@nifty.com

**

Department of Mathematics
University of Gour Banga
P.O. Mokdumpur, Malda 732-103, India
E-mail: spmodak2000@yahoo.co.in